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Homogeneous equations

A function $f(x, y)$ is **homogeneous of degree** n in x and y if

$$f(ax, ay) = a^n f(x, y).$$

Roughly, this means that the "total power" of x and y is the same in all the terms of $f(x, y)$. Here are some examples.

Example. $\sin \dfrac{x}{y}$ is homogeneous of degree 0:

$$\sin \frac{ax}{ay} = \sin \frac{x}{y} = a^0 \sin \frac{x}{y}. \quad \square$$

Example. $\dfrac{2x - 3y}{5x + 4y}$ is also homogeneous of degree 0:

$$\frac{2ax - 3ay}{5ax + 4ay} = \frac{2x - 3y}{5x + 4y} = a^0 \frac{2x - 3y}{5x + 4y}. \quad \square$$

Example. $\cos x$ is not homogeneous of any degree:

$$\cos ax \neq a^n \cos x$$

is not an identity for any n. $\square$

Example. $4x^5 - 7x^3y^2 + xy^4$ is homogeneous of degree 5:

$$4(ax)^5 - 7(ax)^3(ay)^2 + (ax)(ay)^4 = a^5 \left(4x^5_7x^3y^2 + xy^4\right). \quad \square$$

Here is how this applies to differential equations. A first-order equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is **homogeneous** if M and N are homogeneous functions *of the same degree.*

Example. The equation

$$(x^2 - 3xy)\,dx + (7x^2 - y^2)\,dy = 0$$

is homogeneous, since $x^2 - 3xy$ and $7x^2 - y^2$ are homogeneous of degree 2.

On the other hand,

$$(x + 5y)\,dx - (x^2 + 4y^2)\,dy = 0$$

is not homogeneous; $x + 5y$ and $x^2 + 4y^2$ are individually homogeneous, but not of the same degree.

$$(\sin x - \cos y)\, dx + x \cos y\, dy = 0$$

is not homogeneous, since $\sin x - \cos y$ and $x \cos y$ are not homogeneous. $\square$

The following two facts can be used to simplify a homogeneous differential equation.

Fact 1: If M and N are homogeneous of the same degree, then $\dfrac{M}{N}$ is homogeneous of degree 0.

Proof:

$$\frac{M(ax, ay)}{N(ax, ay)} = \frac{a^n M(x, y)}{a^n N(x, y)} = \frac{M(x, y)}{N(x, y)} = a^0 \frac{M(x, y)}{N(x, y)}. \quad \square$$

Fact 2: If f is homogeneous of degree 0, then f can be expressed as a function of $\dfrac{y}{x}$.

Proof: Since f is homogeneous of degree 0, $f(ax, ay) = a^0 f(x, y) = f(x, y)$ is an identity. Set $a = \dfrac{1}{x}$:

$$f\left(1, \frac{y}{x}\right) = f(x, y).$$

The left side is a function of $\dfrac{y}{x}$. $\square$

Now suppose

$$M\, dx + N\, dy = 0$$

is homogeneous. Rewrite it as

$$\frac{dy}{dx} = -\frac{M}{N}.$$

The right side is homogeneous of degree 0 (Fact 1), so it can be written as a function of $\dfrac{y}{x}$ (Fact 2). Suppose then that

$$-\frac{M}{N} = g\left(\frac{y}{x}\right).$$

Let $y = vx$, so $\dfrac{y}{x} = v$. Then

$$-\frac{M}{N} = g(v),$$

and by the Product Rule,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

The original equation becomes

$$v + x\frac{dv}{dx} = g(v) \quad \text{or} \quad \frac{dv}{dx} = \frac{g(v) - v}{x}.$$

This equation can be solved by separation of variables.

Example. Solve $y' = \dfrac{3x - y}{x + y}$.

The right side is clearly homogeneous of degree 0.

Let $y = vx$, so $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substitute:

$$v + x\frac{dv}{dx} = \frac{3-v}{1+v}, \quad x\frac{dv}{dx} = \frac{3-v}{1+v} - v = \frac{(3+v)(1-v)}{1+v}.$$

Separate:

$$\int \frac{1+v}{(3+v)(1-v)}\, dv = \int \frac{dx}{x}.$$

Decompose the integrand on the left using partial fractions:

$$\frac{1+v}{(3+v)(1-v)} = \frac{A}{3+v} + \frac{B}{1-v}$$

$$1 + v = A(1-v) + B(3+v)$$

Setting $x = 1$ yields $2 = 4B$, so $B = \frac{1}{2}$. Setting $x = -3$ yields $-2 = 4A$, so $A = -\frac{1}{2}$. Therefore,

$$\frac{1+v}{(3+v)(1-v)} = \frac{1}{2}\left(-\frac{1}{3+v} + \frac{1}{1-v}\right).$$

Now

$$\int \frac{1}{2}\left(-\frac{1}{3+v} + \frac{1}{1-v}\right) dv = \int \frac{dx}{x}, \quad \frac{1}{2}\left(-\ln|3+v| - \ln|1-v|\right) = \ln|x| + C.$$

Combine the logs on the left, then exponentiate to kill the logs:

$$\ln|(3+v)(1-v)| = -2\ln|x| - 2C, \quad (3+v)(1-v) = \frac{C_0}{x^2}.$$

Finally, put y back:

$$\left(3 + \frac{y}{x}\right)\left(1 - \frac{y}{x}\right) = \frac{C_0}{x^2}, \quad (3x+y)(x-y) = C_0. \quad \square$$

Example. Solve $(x - y\ln y + y\ln x)\, dx + x(\ln y - \ln x)\, dy = 0$.

Rewrite the equation as

$$\left(x - y\ln\frac{y}{x}\right) dx + x\ln\frac{y}{x}\, dy = 0.$$

$x - y\ln\frac{y}{x}$ and $x\ln\frac{y}{x}$ are homogeneous of degree 1.
Rearrange the equation:

$$\frac{dy}{dx} = \frac{y\ln\frac{y}{x} - x}{x\ln\frac{y}{x}}.$$

The right side is homogeneous of degree 0. Let $y = vx$, so $v = \frac{y}{x}$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substitute:

$$v + x\frac{dv}{dx} = \frac{xv\ln v - x}{x\ln v} = \frac{v\ln v - 1}{\ln v}, \quad x\frac{dv}{dx} = \frac{v\ln v - 1}{\ln v} - v = \frac{1}{\ln v}.$$

Separate:

$$\int \ln v\, dv = \int \frac{1}{x}\, dx.$$

Integrate $\ln v$ by parts:

$$\begin{array}{cccc} & \dfrac{d}{dv} & & \int dv \\ + & \ln v & & 1 \\ & & \searrow & \\ - & \dfrac{1}{v} & \rightarrow & v \end{array}$$

Therefore,

$$\int \ln v\, dv = v \ln v - \int \frac{1}{v} \cdot v\, dv = v \ln v - \int dv = v \ln v - v + C.$$

Hence,

$$v \ln v - v = \ln x + C.$$

Put y back:

$$\frac{y}{x} \ln \frac{y}{x} - \frac{y}{x} = \ln x + C, \quad y \ln \frac{y}{x} - y = x \ln x + Cx. \quad \square$$

Example. Solve $(x + y + 1)\, dx + (x + 2y - 3)\, dy = 0$.

This would be homogeneous if the "1" and "3" weren't there. The idea is to make a preliminary substitution

$$x = u + h, \quad y = v + k.$$

I will *choose* h and k so that the result is homogeneous.
Since $dx = du$ and $dy = dv$,

$$(u + v + h + k + 1)\, du + (u + 2v + h + 2k - 3)\, dv = 0.$$

I want to pick h and k so that the constant terms go away:

$$h + k + 1 = 0, \quad h + 2k - 3 = 0.$$

Solving simultaneously, I obtain $k = 4$, $h = -5$. The substitution is

$$x = u - 5, \quad y = v + 4.$$

With this substitution, the equation becomes

$$(u + v)\, du + (u + 2v)\, dv = 0, \quad \text{or} \quad \frac{dv}{du} = -\frac{u + v}{u + 2v}.$$

Let $v = wu$, so $w = \dfrac{v}{u}$ and $\dfrac{dv}{du} = w + u\dfrac{dw}{du}$.
Then

$$w + u\frac{dw}{du} = -\frac{u + wu}{u + 2wu} = -\frac{1 + w}{1 + 2w}, \quad u\frac{dw}{du} = -\frac{1 + w}{1 + 2w} - w = \frac{2w^2 + 2w + 1}{2w + 1}.$$

Separate:

$$\int \frac{2w + 1}{2w^2 + 2w + 1}\, dw = -\int \frac{du}{u}, \quad \frac{1}{2} \ln |2w^2 + 2w + 1| = -\ln |u| + C.$$

Put v bacK;

$$\frac{1}{2} \ln \left| 2\left(\frac{v}{u}\right)^2 + 2\frac{v}{u} + 1 \right| = -\ln |u| + C.$$

Put x and y back:

$$\frac{1}{2}\ln\left|2\left(\frac{y-4}{x+5}\right)^2+2\frac{y-4}{x+5}+1\right|=-\ln|x+5|+C.\quad\square$$

Example. Solve $(x+y+1)\,dx+(2x+2y-1)\,dy=0$.

This looks like the previous problem. But if you let

$$x=u+h,\quad y=v+k,$$

and then try to choose h and k so the constant terms go away, you'll get stuck!

Reason: The h and k equations become

$$h+k=-1,\quad 2h+2k=1,$$

and these equations are inconsistent — there are no solutions.

Instead, let $z=x+y$, so $dz=dx+dy$. Substitute and eliminate x:

$$(z+1)(dz-dy)+(2z-1)\,dy=0,\quad 1-\frac{dz}{dy}=\frac{2z-1}{z+1},\quad -\frac{dz}{dy}=\frac{2z-1}{z+1}-1=\frac{z-2}{z+1}.$$

Separate:

$$-\int\frac{z+1}{z-2}\,dz=\int dy,\quad -z-3\ln|z-2|=y+C.$$

Put x back:

$$-x-y-3\ln|x+y-2|=y+C.\quad\square$$